Proofs,
Carefully Made

A Portfolio of Mathematical Arguments

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Foundations $\longrightarrow$ Real Analysis

Editorial Note

This portfolio was reconstructed from earlier typeset documents whose original source files were no longer available. Course identifiers, grading material, and classroom-specific objectives have been removed. Notation, typography, and organization have been standardized; the mathematical arguments retain the substance of the source documents.

Contents

Part I
Foundations

1 The River Crossing

Theorem 1.1. *A farmer can carry a fox, a goose, and a bag of beans across a river while leaving every purchase intact, provided the boat carries only the farmer and one purchase at a time and neither the fox and goose nor the goose and beans may be left together unattended.*

METHOD: CONSTRUCTIVE PROOF

Proof. First take the goose to the far bank and return alone. Take the beans across, leave them there, and bring the goose back. Leave the goose on the original bank and take the fox across. The fox and beans may remain together safely. Return alone once more and take the goose across. The farmer, fox, goose, and beans are now all on the far bank, and neither prohibited pair was ever left unattended. □

2 An Odd Polynomial

Theorem 2.1. *For every integer m, the integer $6m^3 - 8m + 9$ is odd.*

METHOD: DIRECT PROOF BY PARITY CASES

Proof. If $m = 2k$ is even, then

$$6m^3 - 8m + 9 = 48k^3 - 16k + 9 = 2(24k^3 - 8k + 4) + 1,$$

which is odd. If $m = 2k + 1$ is odd, then

$$6m^3 - 8m + 9 = 48k^3 + 72k^2 + 20k + 7 = 2(24k^3 + 36k^2 + 10k + 3) + 1.$$

This is also odd. Every integer is even or odd, so the conclusion follows. □

3 Closure and Counterexample

Conjecture 3.1. *The set of integers whose decimal representation does not end in zero is closed under addition.*

Disproof. Both 6 and 4 belong to the set, but $6 + 4 = 10$ does not. Thus the set is not closed under addition. □

Theorem 3.2. *The set $F = \{5^n : n \in \mathbb{Z}\}$ is closed under multiplication.*

Proof. Let $x = 5^a$ and $y = 5^b$ be elements of F. Then $xy = 5^{a+b}$. Because $a + b \in \mathbb{Z}$, the product xy also belongs to F. □

4 A Modular Implication

Theorem 4.1. *For every integer a, if $a \equiv 2 \pmod 6$, then $a^2 \equiv 4 \pmod 6$.*

Method: Direct proof

Proof. The hypothesis gives $a = 6k + 2$ for some $k \in \mathbb{Z}$. Hence

$$a^2 - 4 = (6k+2)^2 - 4 = 36k^2 + 24k = 6(6k^2 + 4k).$$

Therefore $6 \mid (a^2 - 4)$, which is exactly $a^2 \equiv 4 \pmod 6$. □

5 Parity of a Cubic Expression

Theorem 5.1. *For every integer z, if $z^3 - 4z$ is even, then z is even.*

Method: Contrapositive

Proof. Suppose z is odd, so $z = 2m + 1$ for some $m \in \mathbb{Z}$. Then

$$z^3 - 4z = (2m+1)^3 - 4(2m+1) = 2(4m^3 + 6m^2 - m - 2) + 1,$$

which is odd. Thus the contrapositive holds, and therefore the original implication holds. □

6 An Irrational Difference

Theorem 6.1. *The number $\sqrt{18} - \sqrt{2}$ is irrational.*

Method: Contradiction

Proof. Since $\sqrt{18} = 3\sqrt{2}$, we have $\sqrt{18} - \sqrt{2} = 2\sqrt{2}$. If this number were rational, division by the nonzero rational number 2 would imply that $\sqrt{2}$ is rational, contradicting the known irrationality of $\sqrt{2}$. □

7 Products Modulo Three

Theorem 7.1. *For integers x and y, $xy \equiv 0 \pmod 3$ if and only if $x \equiv 0 \pmod 3$ or $y \equiv 0 \pmod 3$.*

Method: Biconditional; contrapositive and direct proof

Proof. For the forward direction, prove the contrapositive. If neither x nor y is congruent to 0 modulo 3, then each is congruent to 1 or 2. The four possible products are congruent to $1, 2, 2$, or 1 modulo 3, never 0. Therefore $xy \not\equiv 0 \pmod 3$.

For the reverse direction, if $x \equiv 0 \pmod 3$, write $x = 3k$; then $xy = 3(ky)$, so $xy \equiv 0 \pmod 3$. The same argument applies if $y \equiv 0 \pmod 3$. □

8 Sum of Fibonacci Squares

Theorem 8.1. *If $f_1, f_2, \ldots$ are the Fibonacci numbers, then for every $n \in \mathbb{N}$,*

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}.$$

METHOD: MATHEMATICAL INDUCTION

Proof. For $n = 1$, $f_1^2 = 1 = f_1 f_2$. Assume the identity holds at $n = k$. Then

$$\begin{aligned} f_1^2 + \cdots + f_k^2 + f_{k+1}^2 &= f_k f_{k+1} + f_{k+1}^2 \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1} f_{k+2}. \end{aligned}$$

Thus the identity holds for $k + 1$, and induction completes the proof. □

9 A Proper Subset

Theorem 9.1. *Let $X = \{x \in \mathbb{Z} : x \equiv 2 \pmod 6\}$ and $Y = \{y \in \mathbb{Z} : 3 \mid (y - 5)\}$. Then $X \subsetneq Y$.*

METHOD: ELEMENT CHASING

Proof. If $x \in X$, then $x - 2 = 6k$ for some $k \in \mathbb{Z}$. Hence

$$x - 5 = 6k - 3 = 3(2k - 1),$$

so $x \in Y$. Thus $X \subseteq Y$. Moreover, $5 \in Y$ because $3 \mid 0$, but $5 \notin X$ because $5 \not\equiv 2 \pmod 6$. Therefore $X \subsetneq Y$. □

10 A Surjection That Is Not Injective

Theorem 10.1. *Let $M_2(\mathbb{R})$ be the set of 2×2 real matrices and define*

$$p\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad + bc.$$

Then $p : M_2(\mathbb{R}) \to \mathbb{R}$ is surjective but not injective.

Proof. The matrices with every entry equal to 1 and every entry equal to -1 are distinct, yet both have image 2, so p is not injective. For any $r \in \mathbb{R}$,

$$p\left(\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}\right) = r.$$

Thus every real number has a preimage and p is surjective. □

Part II
Real Analysis

11 An Absolute-Value Inequality

Theorem 11.1. *For integers a, b with $b \neq 0, 1, -1$,*

$$\left|\frac{a+1}{b+1}\right| \leq \frac{|a|+1}{|b|-1}.$$

METHOD: TRIANGLE AND REVERSE-TRIANGLE INEQUALITIES

Proof. The triangle inequality gives $|a+1| \leq |a|+1$. The reverse triangle inequality gives

$$|b+1| \geq \big||b|-1\big| = |b|-1,$$

where the last equality follows from $|b| > 1$. Both denominators are positive, so

$$\left|\frac{a+1}{b+1}\right| = \frac{|a+1|}{|b+1|} \leq \frac{|a|+1}{|b|-1}.$$

□

12 Suprema of Sum Sets

Theorem 12.1. *Let $A, B \subseteq \mathbb{R}$ be nonempty and define $C = \{a+b : a \in A,\ b \in B\}$. If A and B are bounded above, then*

$$\sup C = \sup A + \sup B.$$

In the extended real numbers the identity remains valid when either set is unbounded above.

Proof. Write $\alpha = \sup A$ and $\beta = \sup B$. For every $a \in A$ and $b \in B$, $a + b \leq \alpha + \beta$, so $\alpha + \beta$ is an upper bound for C. Given $\varepsilon > 0$, choose $a \in A$ and $b \in B$ with $a > \alpha - \varepsilon/2$ and $b > \beta - \varepsilon/2$. Then $a + b > \alpha + \beta - \varepsilon$, so no smaller number is an upper bound. Hence $\sup C = \alpha + \beta$. If either set is unbounded above, C is unbounded above as well, giving the extended-real interpretation $+\infty$. □

13 Limit of a Rational Sequence

Theorem 13.1. *The sequence*

$$a_n = \frac{2n+1}{3n^2+5}$$

converges to 0.

METHOD: ε–N PROOF

Proof. For $n \geq 1$,

$$0 < \frac{2n+1}{3n^2+5} \leq \frac{3n}{3n^2} = \frac{1}{n}.$$

Given $\varepsilon > 0$, choose $N > 1/\varepsilon$. Then $n > N$ implies

$$|a_n - 0| \leq \frac{1}{n} < \varepsilon.$$

Therefore $a_n \to 0$. □

14 Reciprocals of a Cauchy Sequence

Theorem 14.1. *Suppose* (a_n) *is a Cauchy sequence of nonzero real numbers and* $a_n \not\to 0$. *Then there exist* N *and* $c > 0$ *such that* $|a_n| \geq c$ *for all* $n \geq N$, *and the sequence* $(1/a_n)$ *is Cauchy.*

Proof. Because $\mathbb{R}$ is complete, (a_n) converges to some L. The assumption gives $L \neq 0$. Choose N_0 so that $n \geq N_0$ implies $|a_n - L| < |L|/2$. Then $|a_n| \geq |L|/2 =: c$.

Given $\varepsilon > 0$, choose $N \geq N_0$ so that $m, n \geq N$ implies $|a_m - a_n| < c^2\varepsilon$. Then

$$\left|\frac{1}{a_n} - \frac{1}{a_m}\right| = \frac{|a_m - a_n|}{|a_n a_m|} \leq \frac{|a_m - a_n|}{c^2} < \varepsilon.$$

Hence $(1/a_n)$ is Cauchy. □

15 Convergence of Weighted Series

Theorem 15.1. *Let* $a_k \geq 0$ *and suppose* $\sum_{k=1}^{\infty} a_k$ *converges. If* $b_k \to L$ *for a finite real number* L, *then* $\sum_{k=1}^{\infty} a_k b_k$ *converges absolutely. In particular this holds whether* L *is positive or negative.*

Proof. Every convergent sequence is bounded, so there is $M > 0$ with $|b_k| \leq M$ for every k. Therefore

$$|a_k b_k| \leq M a_k.$$

Since $\sum M a_k = M \sum a_k$ converges, the comparison test shows that $\sum |a_k b_k|$ converges. Thus $\sum a_k b_k$ converges absolutely. The same argument applies to any finite limit L, including $L < 0$. □

16 Interior, Exterior, and Boundary

Theorem 16.1. *For* $A, B \subseteq \mathbb{R}$:

(a) if A *is open, then* $\operatorname{Int}(A) = A$;

(b) if B *is closed, then* $\operatorname{Ext}(B) = B^c$;

(c) $\partial\mathbb{R} = \partial\varnothing = \varnothing$.

Proof. If A is open, every $x \in A$ has a neighborhood contained in A, so $A \subseteq \operatorname{Int}(A)$. The reverse inclusion follows from the definition of interior.

If B is closed, then B^c is open. Thus every point of B^c has a neighborhood contained in B^c, which says $B^c \subseteq \operatorname{Ext}(B)$. The reverse inclusion is again immediate from the definition.

Finally, every neighborhood of a boundary point of E must meet both E and E^c. No neighborhood can meet both $\mathbb{R}$ and $\varnothing$, so neither $\mathbb{R}$ nor $\varnothing$ has a boundary point. □